Topological Manifolds

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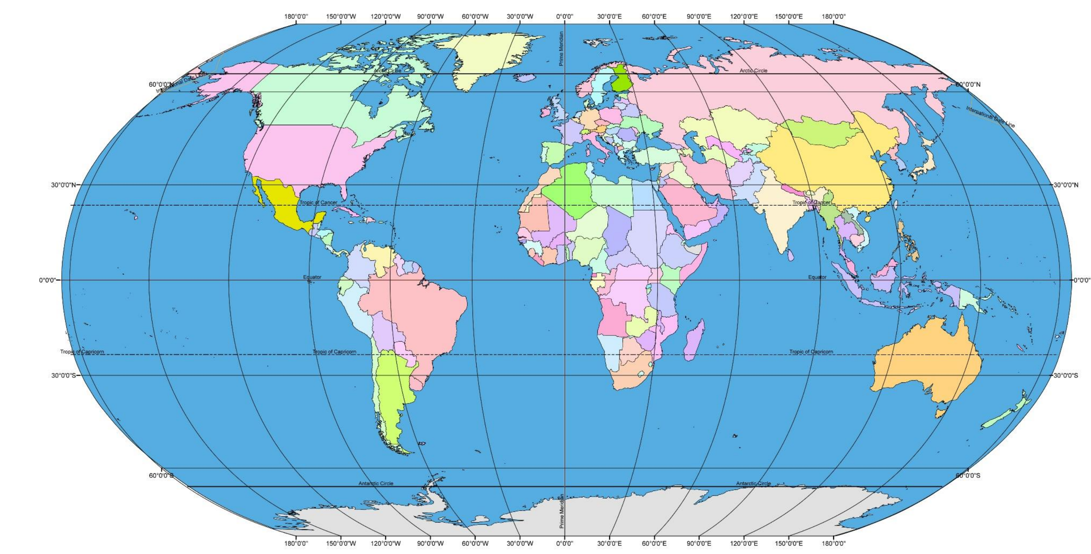


ABSTRACT

Topological Manifolds are abstract spaces that locally resemble Euclidean space. For example, consider a round globe and a flat map. The map is a 2-dimensional representation of a 3-dimensional space. Given any point on the globe we can find a corresponding position on the map, and vice versa. This correspondence is called a chart. With a sufficient number of charts, we can describe the whole space. Such a collection of charts is called an Atlas. It is possible to construct different Atlases for the same space, allowing us to move from one chart, to the space, to another chart. This process is called a transition map.

The areas of focus for this project include several examples of manifolds such as curves, n-spheres, and the torus. We explore and illustrate different approaches to charts on these manifolds, the properties of a manifold, examples of spaces that fail to meet these requirements, and the derivation of transition maps.

Defn. A Real n-dimensional topological manifold is a Hausdorff, Second Countable, topological space which is locally homeomorphic to n-dimensional Euclidean space, i.e. R.

Locally Homeomorphic – Small open sets on a manifold M, resemble Euclidean space. i.e. there exists a homeomorphism f:M → V where V is a subset of R. This function f is called a chart.

A collection of charts that span the entire space M, is called an Atlas.

Example 1: 1-Sphere

$S_1 := \{(x,y): x^2+y^2=1\}$

This is the familiar definition of a the unit circle centered at the origin.

Atlas 1: *(orthographic projection)*

$U = \{\varphi_1, \varphi_2\}$

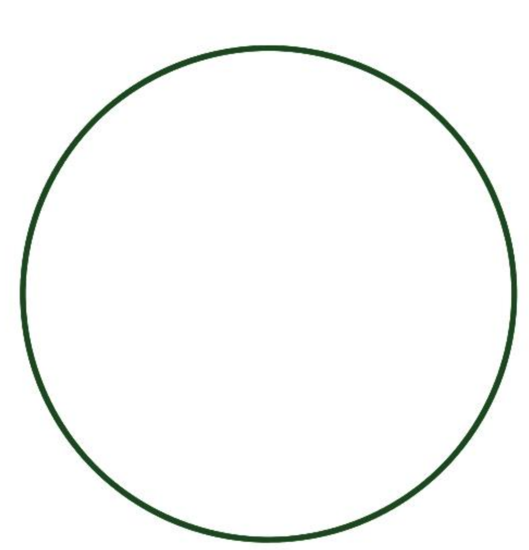
$\varphi_1(x,y) = y; y > 0$

$\varphi_2(x,y) = y; y \le 0$

Atlas 2:

$\theta_1(x,y) = tan^{-1}\left(\frac{y}{x+1}\right); x \neq -1$

$\theta_2(x,y) = tan^{-1}\left(\frac{y}{1-x}\right); x \neq -1$



Example 2: 2-Sphere

$S_2 := \left\{(x,y,z): \sqrt{x^2+y^2+z^2} = 1\right\}$

This is the unit sphere in $\mathbb{R}^2$.

It is an example of a 2-manifold

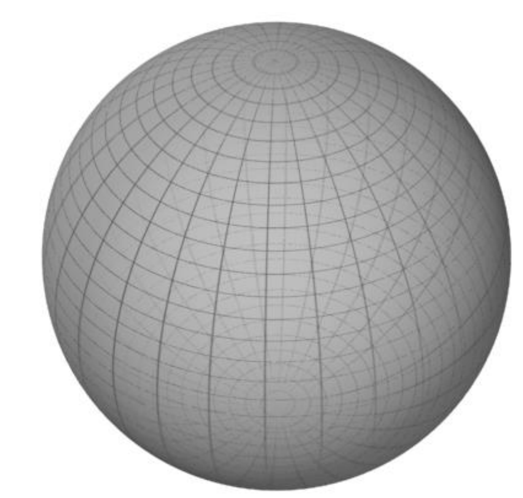
Some examples of Atlases on the unit sphere are:

Orthographic Projection:

$U = \{O_1, O_2\}$

$O_1\big((x,y,z)\big) = (x,y); z > 0$

$O_2\big((x,y,z)\big) = (x,y); z \le 0$



Example 3: N-Sphere

The n-sphere is a collection of points that extends the idea of a sphere to higher dimensions. Although we cannot graphically represent this space we can still talk about its charts and atlases.

Defn. $S_n := \{\boldsymbol{x} = (x_1, x_2, \dots, x_n): ||\boldsymbol{x}|| = 1\}$

Similar to the previous examples in 1, and 2 space, we can form an atlas through orthographic projection.

$U = \{\varphi_1, \varphi_2\}; where$

$\varphi_1(\boldsymbol{x}) = (x_1, x_2, \dots, x_{n-1});\ x_n > 0$

$\varphi_2(\boldsymbol{x}) = (x_1, x_2, \dots, x_{n-1});\ x_n \le 0$

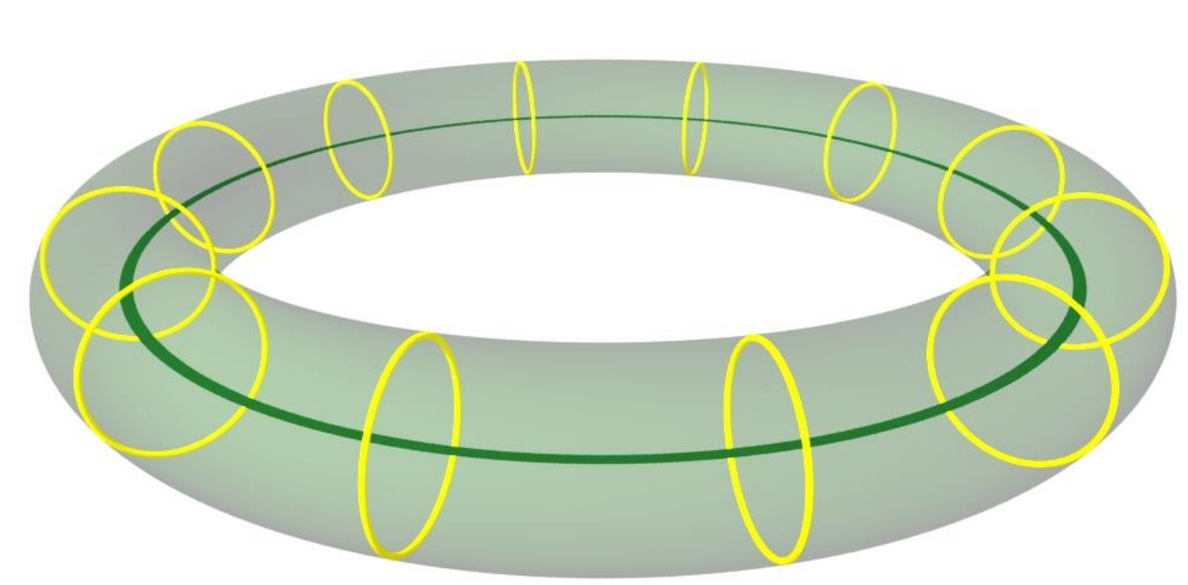
?

Example 4: Torus

Is a 2- manifold.

Defn. $S^1 \times S^1 with\ charts\ \{\varphi_{S_1}, \varphi_{S_1}\}$

TORUS AS CARTESIAN PRODUCT



$\mathcal{U} = \{\phi_1, \phi_2\}$

$\phi_1 : (x,y) \to x, y \ge 0 \quad \phi_1^{-1} : n \to \left(n, \sqrt{1-n^2}\right)$

$\phi_2 : (x,y) \to x, y < 0 \quad \phi_2^{-1} : n \to \left(n, -\sqrt{1-n^2}\right)$

$\theta_1 : (x,y) \to \tan^{-1}\left(\frac{x+1}{y}\right), (x,y) \neq (-1,0) \quad \theta_1^{-1} : n \to$

$\theta_2 : (x,y) \to \tan^{-1}\left(\frac{1-x}{y}\right), (x,y) \neq (1,0)$

Consider a number $n \in \mathbb{R}$ *on the map of* ϕ_1. *n coorisponds* to *a unique point p on the circle.* $\theta_1(p) \to n' \in \mathbb{R}$. *The point p lies on both maps, We can transition* from *one* to *another using a* ***transition map.***

Transition map :

$t_1 : \mathbb{R} \to \mathbb{R}$

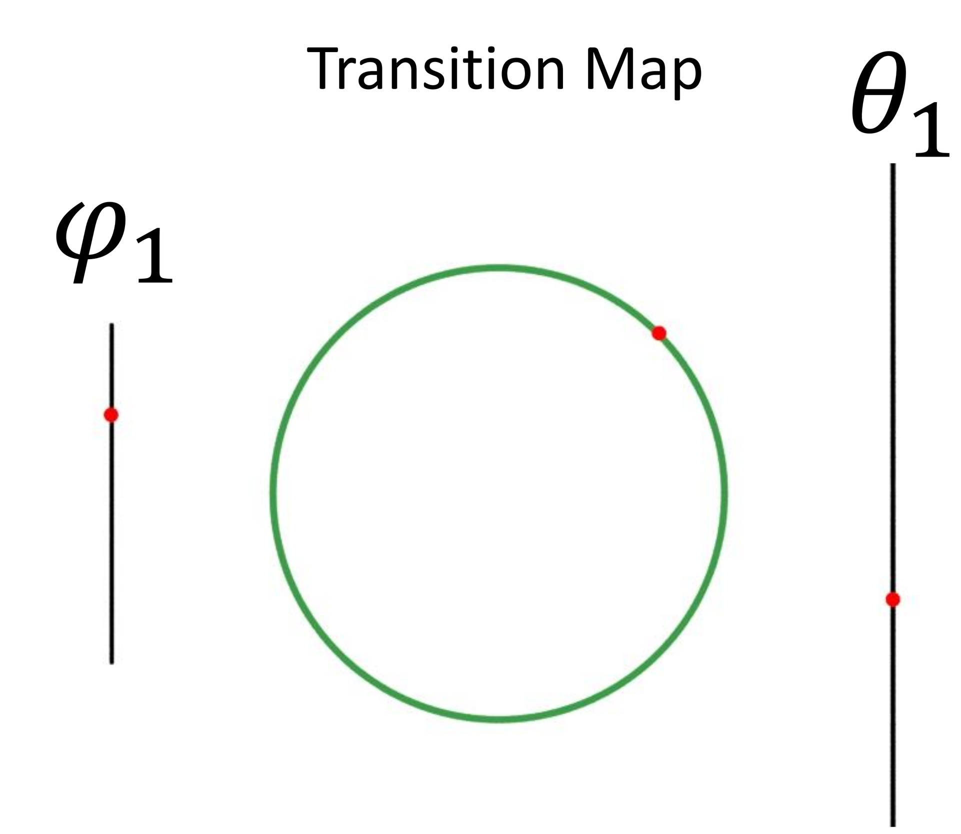
$n \to \theta_1\left(\phi_1^{-1}(n)\right) = \theta_1\left(\left(n, \sqrt{1-n^2}\right)\right) = \tan^{-1}\left(\frac{n}{\sqrt{1-n^2}}\right)$

Example :

Consider $n = \frac{1}{\sqrt{2}}$ *that corrisponds* $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$t_1(n) = t_1\left(\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}}\right) = \left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = 1$

Transition Map

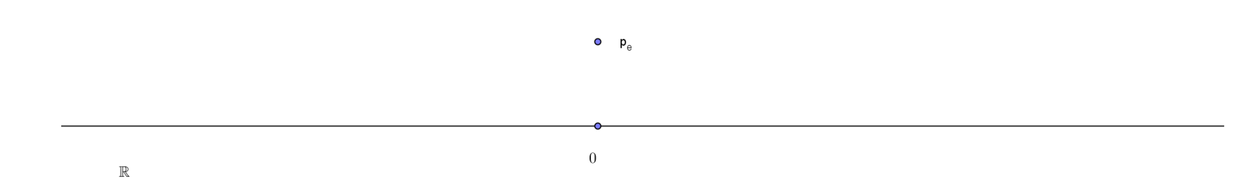


Why Hausdorff?

The Hausdorff property is required to prevent unusual spaces qualifying as manifolds.

Consider the topological space:

(X , T). ;where X = $\mathbb{R}\ \cup\ p_e$ where p_e is an extra point not in $\mathbb{R}$. And $\mathcal{T} = \{(a,b): a < b\} \cup \{(c,0) \cup \{p_e\} \cup (0,d): c < 0, d > 0$

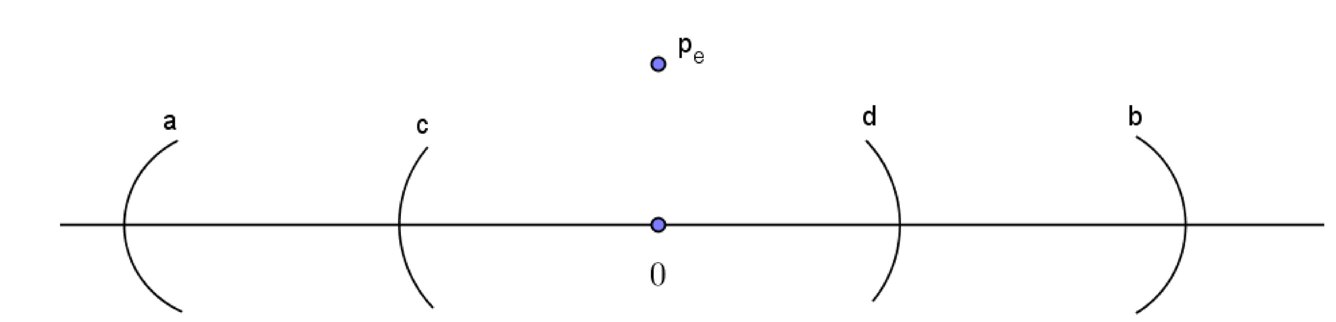


This space is locally homeomorphic to $\mathbb{R}\ and$ is 2nd countable, but it is not Haudorff.

Consider points p_e amd 0. Can we find two distinct open sets U, V such that $p_e \in U\ and\ 0 \in V$?

All neighborhoods of 0 are of the form (a,b) where a < 0 and b > 0.

All neighborhoods of p_e are of the form $(c,0) \cup \{p_e\} \cup (0,d)$.



Clearly every configuration of U and V will have a non empty intersection.

Therefore (X,$\mathcal{T}$) $is\ not\ a\ manifold$.

References

-Foundations of Topology: C. Wayne Patty; 2nd edition

-Geometry and Toplogy Marco Guarltieri

-Introduction to Toplogy: Pure and Applied: Adams Franzosa