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Inside Out: Properties of the Klein Bottle

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Inside Out: Properties of the Klein Bottle

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Abstract:

A Klein Bottle is a two-dimensional manifold in mathematics that, despite appearing like an ordinary bottle, is actually completely closed and completely open at the same time. The Klein Bottle, which can be represented in three dimensions with self-intersection, is a four-dimensional object with no intersection of material. In this presentation we illustrate some topological properties of the Klein Bottle, use the Möbius Strip to help demonstrate the construction of the Klein Bottle, and use mathematical properties to show that the Klein Bottle intersection that appears in $\mathbb{R}^3$ does not exist in $\mathbb{R}^4$.

Introduction:

Topology is the mathematical study of objects with properties that stay with the object through deformations, twistings, and stretchings. The Möbius Strip, also called the Möbius Band, is a simple one-sided surface that is easy to make with a strip of paper. By taking the ends of the strip, twisting one of them 180°, and joining them together, it will create this famous one-sided surface. The Möbius Strip was first introduced by an astronomer and mathematician named August Ferdinand Möbius. When two Möbius Strips are glued along their edges, it creates a surface equivalent to the Klein Bottle. German mathematician Christian Felix Klein first described the Klein Bottle in 1882.

A Klein Bottle cannot be properly constructed in 3 dimensions (i.e. it intersects itself), but you can still see the shape in 3-D with an intersection (figures 1 and 4).

$\mathbb{R}$ refers to the set of all real numbers, with $\mathbb{R}^3$ and $\mathbb{R}^4$ being the third and fourth dimensions, respectively.

Objective:

To provide people, who may not be familiar with topology, a view into how the Möbius Strip (a relatively well-known construction) and the Klein Bottle relate to one another using topology, and to help the reader visualize the Klein Bottle in the fourth dimension.

Results:

First, we will illustrate that the Klein Bottle does not intersect itself in $\mathbb{R}^4$. Before doing so, we will step back to dimensions we can see. In figure 2, we see a two-dimensional shape, with length and width, that has an intersection with itself. If we bring that shape into three-dimensions, we can lift one of the lines up to give the shape height, and thus make it so the shape does not intersect itself. We can apply that same logic to $\mathbb{R}^3$ and $\mathbb{R}^4$.

By figures 1 and 4, we see that when in three dimensions, the Klein Bottle intersects itself once. If the coordinates of $\mathbb{R}^4$ are (x,y,z,w), then in $\mathbb{R}^3$, they would be (x,y,z,0), zero height in the fourth dimension. To remove the intersection in $\mathbb{R}^3$, we need to increase the fourth coordinate (w) to lift the Klein Bottle so it does not intersect itself. (Shiga 56-57)

Below are some of the topological properties of our two constructs:

Topological Properties

Möbius Strip	Klein Bottle
- one side and one boundary	- one side and no boundary
- non-orientable	- non-orientable
- connected	- connected
- closed	- closed

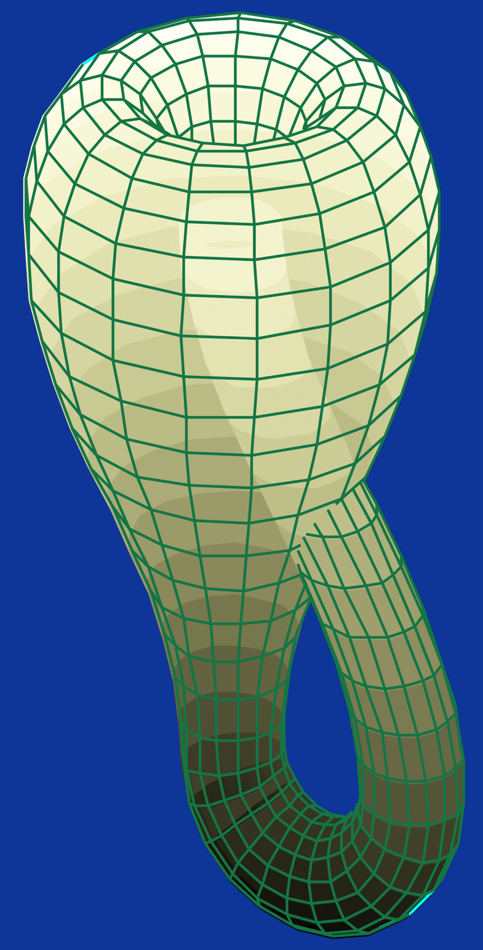


Fig. 1
Tttrung. A Two-dimensional Representation of the Klein Bottle Immersed in Three-dimensional Space. Digital image. Wikipedia. Wikimedia Foundation, n.d. Web. 2 Apr. 2015.

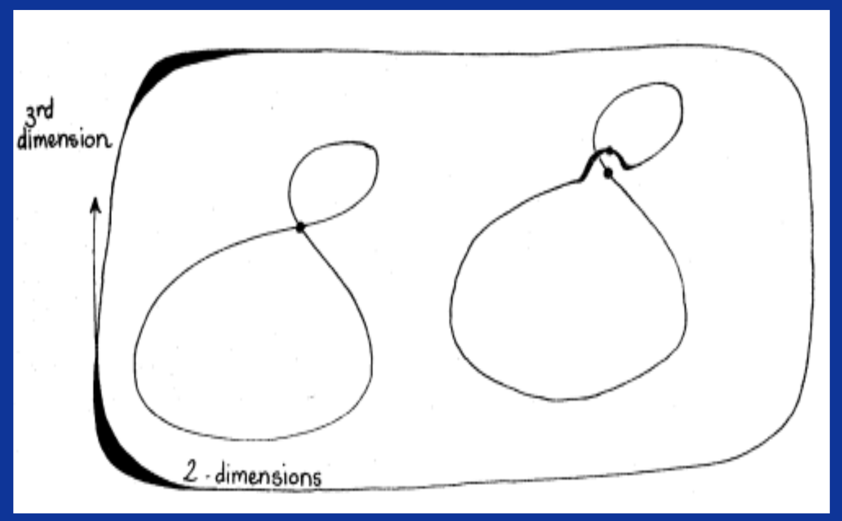


Fig. 2
Zeeman, E. C. Illustration of Intersections Disappearing in the Third Dimension. Digital image. School of Mathematics. University of Edinburgh, n.d. Web. 2 Apr. 2015. Image found on pg. 7.

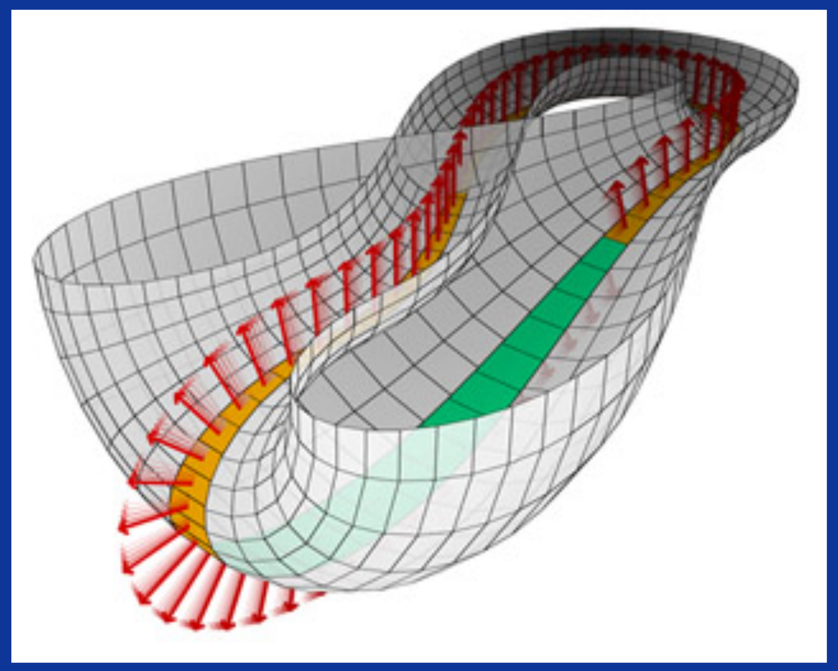


Fig. 3
Polthier, Konrad. *Moebius Strip on a Klein Bottle*. Digital image. *Vismath*. N.p., 2012. Web. 2 Apr. 2015.

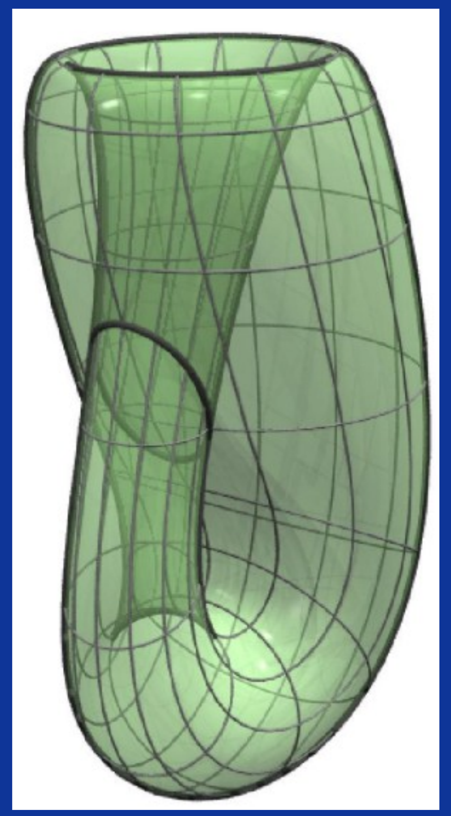


Fig. 4
M. Visual Illusion Klein Bottle. Digital image. OPTIC OPTIC. N.p., n.d. Web. 2 Apr. 2015.

Results (cont.):

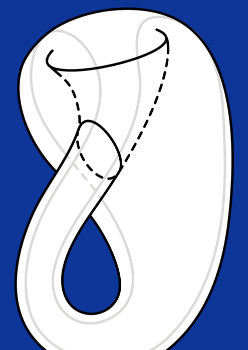
To construct a Möbius Strip in $\mathbb{R}^3$, the following parametric equations are used:

Jerry. Mobius-Strip. Digital image. Mobius Strip: A Young Scientist. N.p., 8 Aug. 2010. Web. 2 Apr. 2015.

$$x(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u$$
$$y(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u$$
$$z(u,v) = \frac{v}{2}\sin\frac{u}{2}$$

To construct a Klein Bottle in $\mathbb{R}^4$, the parametric equations listed here are the most common way to generate this object:

$$x = R\left(\cos\frac{\theta}{2}\cos v - \sin\frac{\theta}{2}\sin 2v\right)$$
$$y = R\left(\sin\frac{\theta}{2}\cos v - \cos\frac{\theta}{2}\sin 2v\right)$$
$$z = P\cos\theta(1 + \epsilon\sin v)$$
$$w = P\sin\theta(1 + \epsilon\sin v)$$



Vierkantswortel2. Surface of Klein Bottle with Traced Line. Digital image. Wikimedia Commons. Wikimedia Foundation, 20 June 2013. Web. 2 Apr. 2015.

(Where R and P are constants that determine aspect ratio, $0 \le \theta < 2\pi$, $0 \le v < 2\pi$, and any small constant ϵ.)

Did You Know?

- The Klein Bottle is actually not a bottle at all! It's a one-sided object that exists in the fourth dimension — you could never fill it up with liquid!
- The Möbius Strip is often used in the construction of conveyer belts and typewriter tape, because using both sides evenly allows for longer usage and more even wear of these items!
- Two Möbius Strips can be combined together to form a Klein Bottle! If you split a Klein Bottle along its line of symmetry, you turn it back into the same two Möbius Strips!
- This fact inspired the mathematician Leo Moser to compose a limerick:

A mathematician named Klein
Thought the Möbius Band was divine.
Said he: "If you glue
The edges of two,
You'll get a weird bottle like mine."

- The Klein Bottle was originally named the "Klein Surface", but was mistranslated due to the similarity between the German spelling of surface (flache) versus bottle (flasche).
- Topologically, the edge of a Möbius Strip is the same as a circle!
- In music theory, mapping out the dyads (groups of two-note chords) gives us the shape of a Möbius Strip! Math can be useful in many fields!
- Six colors are sufficient to color any map on the surface of the Klein Bottle and Möbius Strip.

Conclusions:

We see that both the Möbius Strip and Klein Bottle have interesting and useful geometric and topological properties, and that the Klein Bottle does not actually intersect itself in the fourth dimension.

Though the Klein Bottle is made of two joined Möbius Strips, they do not have identical topological properties.

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